D Wave Heavy Mesons

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We first extract the binding energy $\bar{\Lambda}$ and decay constants of the D wave heavy meson doublets $(1^-, 2^-)$ and $(2^-, 3^-)$ with QCD sum rule in the leading order of heavy quark effective theory. Then we study their pionic (π, K, η) couplings using the light cone sum rule, from which the parameter $\bar{\Lambda}$ can also be extracted. We then calculate the pionic decay widths of the strange/non-strange D wave heavy D/B mesons and discuss the possible candidates for the D wave charm-strange mesons. Further experimental information, such as the ratio between $D_s\eta$ and DK modes, will be very useful to distinguish various assignments for $D_{sJ}(2860, 2715)$.

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I. INTRODUCTION

Recently BarBar reported two new D_s states, $D_{sJ}(2860)$ and $D_{sJ}(2690)$ in the DK channel. Their widths are $\Gamma = 48 \pm 7 \pm 10$ MeV and $\Gamma = 112 \pm 7 \pm 36$ MeV respectively [1]. For $D_{sJ}(2860)$ the significance of signal is 5σ in the D^0K^+ channel and $2.8\ \sigma$ in the $D^+K^0_s$ channel. Belle observed another state $D_{sJ}(2715)$ with $J^P = 1^-$ in $B^+ \to \bar{D}^0 D_{sJ} \to \bar{D}^0 D^0 K^+$ [2]. Its width is $\Gamma = 115 \pm 20$ MeV. No D^*K or $D_s\eta$ mode has been detected for all of them.

The J^P of $D_{sJ}(2860)$ and $D_{sJ}(2690)$ can be $0^+, 1^-, 2^+, 3^-, \cdots$ since they decay to two pseduscalar mesons. $D_{sJ}(2860)$ was proposed as the first radial excitation of $D_{sJ}(2317)$ based on a coupled channel model [3] or the first radial excitation of the ground 1^- state D_s^* using an improved potential model [4]. Colangelo et al considered $D_{sJ}(2860)$ as the D wave 3^- state [5]. The mass of $D_{sJ}(2715)$ or $D_{sJ}(2690)$ is consistent with the potential model prediction of the $c\bar{s}$ radially excited 2^3S_1 state [4, 6]. Based on chiral symmetry consideration, a D wave 1^- state with mass $M = 2720$ MeV is also predicted if the $D_{sJ}(2536)$ is taken as the P wave 1^+ state [7]. The strong decay widths for these states are discussed using the $^3P_0$ model in [8].

The heavy quark effective theory (HQET) provides a systematic expansion in terms of $1/m_Q$ for hadrons containing a single heavy quark, where m_Q is the heavy quark mass [9]. In HQET the heavy mesons can be grouped into doublets with definite j_ℓ^P since the angular monument of light components j_ℓ is a good quantum number in the $m_Q \to \infty$ limit. They are $\frac{1}{2}^-$ doublet $(0^-, 1^-)$ with orbital angular monument $L = 0$, $\frac{1}{2}^+$ doublet $(0^+, 1^+)$ and $\frac{3}{2}^+$ doublet $(1^+, 2^+)$ with $L = 1$. For $L = 2$ there are $(1^-, 2^-)$ and $(2^-, 3^-)$ doublets with $j_\ell^P = \frac{3}{2}^-$and $\frac{5}{2}^-$ respectively. The states with the same J^P, such as the two 1^- and two 1^+ states, can be distinguished in the $m_Q \to \infty$ limit, which is one of the advantage of working in HQET. The D wave heavy mesons $((B_1^{*'}, B_2^*), (B_2^{*'}, B_3))$ were considered in the quark model [10]. The semileptonic decay of B meson to the D wave doublets was calculated using three point QCD sum rule [11]. The decay property of the heavy mesons till $L = 2$ was calculated using the $^3P_0$ model in [12].

Light cone QCD sum rule (LCQSR) has proven very useful in extracting the hadronic form factors and coupling constants in the past decade [13]. Unlike the traditional SVZ sum rule [14], it was based on the twist expansion on the light cone. The strong couplings and semileptonic decay form factors of the low lying heavy mesons have been calculated using this method both in full QCD and in HQET [15].

In this paper we first extract the mass parameters and decay constants of D wave doublets in section II. Then we study the strong couplings of the D wave heavy doublets with light pseduscalar mesons π, K and η in section III. We work in the framework of LCQSR in the leading order of HQET. We present our numerical analysis in section IV. In section V we calculate the strong decay widths to light hadrons and discuss the possible D wave charm-strange heavy meson candidates. The results are summarized in section VI.

II. TWO-POINT QCD SUM RULES

The proper interpolating currents $J^{\alpha_1\cdots\alpha_j}_{j,P,j_\ell}$ for the states with the quantum number j, P, j_ℓ in HQET were given in [16], with j the total spin of the heavy meson, P the parity and j_ℓ the angular momentum of light components. In the $m_Q \to \infty$ limit, the currents satisfy the following conditions

$$
\begin{aligned}
&\langle 0|J^{\alpha_1\cdots\alpha_j}_{j,P,j_\ell}(0)|j', P', j_\ell^{'}\rangle = f_{Pj_l}\delta_{jj'}\delta_{PP'}\delta_{j_\ell j_\ell^{'}}\eta^{\alpha_1\cdots\alpha_j} \ , \\
&i\,\langle 0|T\left(J^{\alpha_1\cdots\alpha_j}_{j,P,j_\ell}(x)J^{\dagger\beta_1\cdots\beta_{j'}}_{j',P',j_\ell^{'}}(0)\right)|0\rangle = \delta_{jj'}\delta_{PP'}\delta_{j_\ell j_\ell^{'}} \\
&\times (-1)^j\,\mathcal{S}\,g_t^{\alpha_1\beta_1}\cdots g_t^{\alpha_j\beta_j}\int dt\delta(x - vt)\,\Pi_{P,j_\ell}(x) \ ,
\end{aligned}
\tag{1}
$$

where $\eta^{\alpha_1\cdots\alpha_j}$ is the polarization tensor for the spin j state. v denotes the velocity of the heavy quark. The transverse metric tensor $g_t^{\alpha\beta} = g^{\alpha\beta} - v^\alpha v^\beta$. $\mathcal{S}$ denotes symmetrizing the indices and subtracting the trace terms separately in the sets $(\alpha_1 \cdots \alpha_j)$ and $(\beta_1 \cdots \beta_j)$. f_{P,j_ℓ} is a constant. Π_{P,j_ℓ} is a function of x. Both of them depend only on P and j_ℓ.

The interpolating currents are [16]

$$J^{\dagger\alpha}_{1,-,\frac{3}{2}} = \sqrt{\frac{3}{4}}\, \bar{h}_v(-i)\left(\mathcal{D}^\alpha_t - \frac{1}{3}\gamma^\alpha_t \not{\mathcal{D}}_t\right) q \ , \tag{2}$$

$$J^{\dagger\alpha_1,\alpha_2}_{2,-,\frac{3}{2}} = \sqrt{\frac{1}{2}}\, T^{\alpha_1,\alpha_2;\,\beta_1,\beta_2}\bar{h}_v(-i) \times \left(\mathcal{D}_{t\beta_1}\mathcal{D}_{t\beta_2} - \frac{2}{5}\mathcal{D}_{t\beta_1}\gamma_{t\beta_2}\not{\mathcal{D}}_t\right) q \ , \tag{3}$$

$$J^{\dagger\alpha_1,\alpha_2}_{2,-,\frac{5}{2}} = -\sqrt{\frac{5}{6}}\, T^{\alpha_1,\alpha_2;\,\beta_1,\beta_2}\bar{h}_v\gamma^5 \times \left(\mathcal{D}_{t\beta_1}\mathcal{D}_{t\beta_2} - \frac{2}{5}\mathcal{D}_{t\beta_1}\gamma_{t\beta_2}\not{\mathcal{D}}_t\right) q \ , \tag{4}$$

$$J^{\dagger\alpha,\beta,\lambda}_{3,-,\frac{5}{2}} = -\sqrt{\frac{1}{2}}\, T^{\alpha,\beta,\lambda;\,\mu,\nu,\sigma}\bar{h}_v\gamma_{t\mu}\mathcal{D}_{t\nu}\mathcal{D}_{t\sigma} q \ , \tag{5}$$

$$J^{\dagger\alpha}_{1,-,\frac{1}{2}} = \sqrt{\frac{1}{2}}\, \bar{h}_v\gamma^\alpha_t q \ , \quad J^{\dagger}_{0,-,\frac{1}{2}} = \sqrt{\frac{1}{2}}\, \bar{h}_v\gamma_5 q \ , \tag{6}$$

$$J^{\dagger\alpha}_{1,+,\frac{1}{2}} = \sqrt{\frac{1}{2}}\, \bar{h}_v\gamma^5\gamma^\alpha_t q \ , \tag{7}$$

$$J^{\dagger\alpha}_{1,+,\frac{3}{2}} = \sqrt{\frac{3}{4}}\, \bar{h}_v\gamma^5(-i)\left(\mathcal{D}^\alpha_t - \frac{1}{3}\gamma^\alpha_t \not{\mathcal{D}}_t\right) q \ , \tag{8}$$

$$J^{\dagger\alpha_1,\alpha_2}_{2,+,\frac{3}{2}} = \sqrt{\frac{1}{2}}\, \bar{h}_v\frac{(-i)}{2} \times \left(\gamma^{\alpha_1}_t\mathcal{D}^{\alpha_2}_t + \gamma^{\alpha_2}_t\mathcal{D}^{\alpha_1}_t - \frac{2}{3}g^{\alpha_1\alpha_2}_t\not{\mathcal{D}}_t\right) q \ , \tag{9}$$

where h_v is the heavy quark field in HQET and $\gamma_{t\mu} = \gamma_\mu - v_\mu \not{v}$. The definitions of $T^{\alpha,\beta;\,\mu,\nu}$ and $T^{\alpha,\beta,\lambda;\,\mu,\nu,\sigma}$ are given in Appendix A.

We first study the two point sum rules for the $(1^-, 2^-)$ and $(2^-, 3^-)$ doublets. We consider the following correlation functions:

$$i\int d^4x\, e^{ikx}\langle\pi|T(J^\alpha_{1,-,\frac{3}{2}}(x)J^\beta_{1,-,\frac{3}{2}})|0\rangle = -g^{\alpha\beta}_t\Pi_{-,\frac{3}{2}}(\omega) \ , \tag{10}$$

$$i\int d^4x e^{ikx}\langle\pi|T(J^{\alpha_1\alpha_2}_{2,-,\frac{5}{2}}(x)J^{\beta_1\beta_2}_{2,-,\frac{5}{2}})|0\rangle = \frac{1}{2}(g^{\alpha_1\beta_1}_t g^{\alpha_2\beta_2}_t + g^{\alpha_1\beta_2}_t g^{\alpha_2\beta_1}_t - \frac{2}{3}g^{\alpha_1\alpha_2}_t g^{\beta_1\beta_2}_t)\Pi_{-,\frac{5}{2}}(\omega) \ , \tag{11}$$

where $\omega = 2v\cdot k$.

At the hadron level,

$$\Pi_{P,j\,l} = \frac{f^2_{P,j_l}}{2\bar{\Lambda}_{P,j_l} - \omega} + \cdots \ .$$

At the quark-gluon level it can be calculated with the leading order lagrangian in HQET. Invoking quark-hadron duality and making the Borel transformation, we get the following sum rules from eqs. (10) and (11)

$$f^2_{-,\frac{3}{2}}\exp\left[-\frac{2\bar{\Lambda}_{-,\frac{3}{2}}}{T}\right] = \frac{1}{2^6\pi^2}\int_0^{\omega_c}\omega^4 e^{-\omega/T}d\omega + \frac{1}{16}\,m_0^2\,\langle\bar{q}q\rangle - \frac{1}{2^5}\langle\frac{\alpha_s}{\pi}G^2\rangle T,$$

$$f^2_{-,\frac{5}{2}}\exp\left[-\frac{2\bar{\Lambda}_{-,\frac{5}{2}}}{T}\right] = \frac{1}{2^7\cdot 5\pi^2}\int_0^{\omega_c}\omega^6 e^{-\omega/T}d\omega + \frac{1}{120}\langle\frac{\alpha_s}{\pi}G^2\rangle T^3 \ . \tag{12}$$

Here $m_0^2\,\langle\bar{q}q\rangle = \langle\bar{q}g\sigma_{\mu\nu}G^{\mu\nu}q\rangle$. Only terms of the lowest order in α_s and operators with dimension less than six have been included. For the $\frac{5}{2}^-$ doublet there is no mixing condensate due to the higher derivation.

We use the following values for the QCD parameters: $\langle\bar{q}q\rangle = -(0.24\text{ GeV})^3$, $\langle\alpha_s GG\rangle = 0.038\text{ GeV}^4$, $m_0^2 = 0.8\text{ GeV}^2$.

Requiring that the high-order power corrections is less than 30% of the perturbation term without the cutoff ω_c and the contribution of the pole term is larger than 35% of the continuum contribution given by the perturbation integral in the region $\omega > \omega_c$, we arrive at the stability region of the sum rules $\omega_c = 3.2 - 3.6$ GeV, $T = 0.8 - 1.0$ GeV.

The results for $\bar{\Lambda}$'s are

$$\bar{\Lambda}_{-,\frac{3}{2}} = 1.42 \pm 0.08\ \text{ GeV} \ , \tag{13}$$

$$\bar{\Lambda}_{-,\frac{5}{2}} = 1.38 \pm 0.09\ \text{ GeV} \ . \tag{14}$$

The errors are due to the variation of T and the uncertainty in ω_c. In Fig. 1 and 2, we show the variations of masses with T for different ω_c.

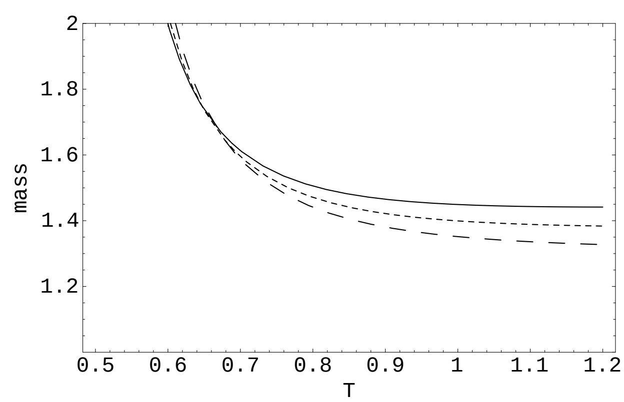

FIG. 1: The variation of the mass parameter $\bar{\Lambda}_{-,\frac{3}{2}}$ with the threshold ω_c (in unit of GeV) for the $\frac{3}{2}^-$ doublet. The long-dashed, short dashed and solid curves correspond to ω_c=3.2, 3.4, 3.6 GeV respectively.

The masses of D wave mesons in the quark model are around 2.8 GeV for D meson and 6 GeV for B meson [10]. $1/m_Q$ correction may be quite important for D wave

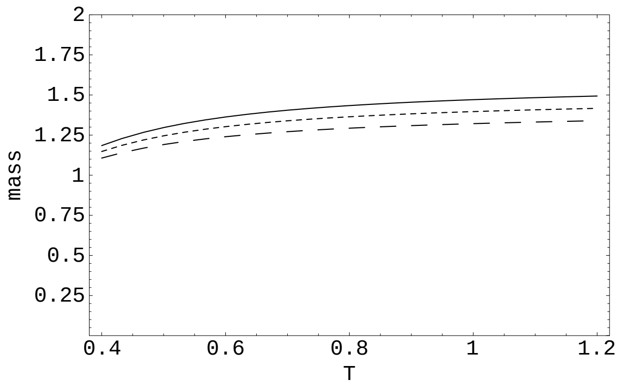

FIG. 2: The variation of the mass parameter $\bar{\Lambda}_{-,\frac{5}{2}}$ with the threshold ω_c (in unit of GeV) for the $\frac{5}{2}^-$ doublet. The long-dashed, short dashed and solid curves correspond to ω_c=3.2, 3.4, 3.6 GeV respectively.

heavy mesons, which will be investigated in a subsequent work.

In the following sections we also need the values of f's:

$$f_{-,\frac{3}{2}} \;=\; 0.39 \pm 0.03 \text{ GeV}^{5/2} \;, \tag{15}$$

$$f_{-,\frac{5}{2}} \;=\; 0.33 \pm 0.04 \text{ GeV}^{7/2} \;. \tag{16}$$

III. SUM RULES FOR DECAY AMPLITUDES

Now let us consider the strong couplings of D wave doublets $\frac{3}{2}^-$ and $\frac{5}{2}^-$ with light hadrons. For the light quark being a u (or d) quark, the D wave heavy mesons decay to pion, while for the light quark being a strange quark, it can decay either to BK or $B_s\eta$. In the following we will denote the light hadron as pion and discuss all three cases. The strong decay amplitudes for D wave 1^- and 3^- states to ground doublet $(0^-, 1^-)$ are

$$\begin{aligned}
&M(B_1^{*'} \to B\pi) = I\epsilon^{\mu} q_{t\mu} g(B_1^{*'}B) \;, \\
&M(B_1^{*'} \to B^*\pi) = I\, i\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu}\epsilon_{\nu}^{'*} v_{\rho} q_{t\sigma} g(B_1^{*'}B^*) \;, \\
&M(B_3 \to B\pi) = I\;\epsilon^{\alpha\beta\lambda}(q_{t\alpha}q_{t\beta}q_{t\lambda} - \frac{1}{6}q_t^2(g_{t\alpha\beta}q_{t\lambda} \\
&\qquad\qquad + g_{t\alpha\lambda}q_{t\beta} + \frac{4}{3}g_{t\beta\lambda}q_{t\alpha}))g(B_3 B) \;, \\
&M(B_3 \to B^*\pi) = I\; i\epsilon^{\mu\nu\sigma\alpha}\;\epsilon_{\alpha\beta\lambda}\,\epsilon_{\mu}^{*}\,v_{\sigma} \;, \\
&\qquad\qquad \times\left[q_{t\nu}\,q_t^{\beta}\,q_t^{\lambda} - \frac{1}{6}q_t^2\left(g_{t\nu}^{\beta}q_t^{\lambda} + g_{t\nu}^{\lambda}q_t^{\beta} \right.\right. \\
&\qquad\qquad \left.\left. + \frac{4}{3}g_t^{\beta\lambda}q_{t\nu}\right)\right] g(B_3 B^*) \;,
\end{aligned} \tag{17}$$

where $\epsilon^{\alpha\beta\lambda}$, ϵ^{μ}, and $\epsilon_{\nu}^{'}$ are polarizations of the D wave 3^-, 1^- states and ground 1^- state respectively. $I = 1, \frac{1}{\sqrt{2}}$ for charged and neutral pion mesons respectively, while for the K and η mesons it equals one. $g(B_1^{*'}B)$ etc are the coupling constants in HQET and are related to those in full QCD by

$$g^{\text{full QCD}}(B_1^{*'}B) = \sqrt{m_{B_1^{*'}} m_B}\; g^{\text{HQET}}(B_1^{*'}B) \;. \tag{18}$$

Because of the heavy quark symmetry, the coupling constants in eq. (17) satisfy

$$\begin{aligned}
g(B_1^{*'}B) \;&=\; g(B_1^{*'}B^*) \;, \\
g(B_3^* B) \;&=\; g(B_3^* B^*) \;.
\end{aligned} \tag{19}$$

In order to derive the sum rules for the coupling constants we consider the correlators

$$\int d^4x\; e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha}_{1,-,\frac{3}{2}}(x)J^{\dagger}_{0,-,\frac{1}{2}}(0)\right)|0\rangle = q_t^{\alpha} I\; G_1(\omega,\omega') \;, \tag{20}$$

$$\int d^4x\; e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha}_{1,-,\frac{3}{2}}(x)J^{\dagger\beta}_{1,+,\frac{1}{2}}(0)\right)|0\rangle = (q_t^{\alpha}q_t^{\beta} - \frac{1}{3}g_t^{\alpha\beta}q_t^2) I\; G_2(\omega,\omega') \;, \tag{21}$$

$$\int d^4x\; e^{ik\cdot x}\langle\pi(q)|T\left(J^{\beta}_{1,+,\frac{3}{2}}(x)J^{\dagger\alpha}_{1,-,\frac{3}{2}}(0)\right)|0\rangle = (q_t^{\alpha}q_t^{\beta} - \frac{1}{3}g_t^{\alpha\beta}q_t^2) I\; G_3^d(\omega,\omega') + g_t^{\alpha\beta} I G_3^s \;, \tag{22}$$

$$\begin{aligned}
\int d^4x\; e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha}_{1,-,\frac{3}{2}}(x)J^{\dagger\alpha_1\alpha_2}_{2,-,\frac{3}{2}}(0)\right)|0\rangle &= \left[\frac{1}{2}(g_t^{\alpha\alpha_1}q_t^{\alpha_2} + g_t^{\alpha\alpha_2}q_t^{\alpha_1}) - \frac{1}{3}g_t^{\alpha_1\alpha_2}q_t^{\alpha}\right] I\; G_4^p(\omega,\omega') \\
&+ \left[q_t^{\alpha}q_t^{\alpha_1}q_t^{\alpha_2} - \frac{1}{6}q_t^2(g_t^{\alpha\alpha_1}q_t^{\alpha_2} + g_t^{\alpha\alpha_2}q_t^{\alpha_1} + \frac{4}{3}g_t^{\alpha_1\alpha_2}q_t^{\alpha})\right] I\; G_4^f(\omega,\omega') \;,
\end{aligned} \tag{23}$$

for the $j_\ell^P = \frac{3}{2}^-$ doublet;

$$\int d^4x\; e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha\beta\lambda}_{3,-,\frac{5}{2}}(x)J^{\dagger}_{0,-,\frac{1}{2}}(0)\right)|0\rangle = \left[q_t^{\alpha}q_t^{\beta}q_t^{\lambda} - \frac{1}{6}q_t^2(g_t^{\alpha\beta}q_t^{\lambda} + g_t^{\alpha\lambda}q_t^{\beta} + \frac{4}{3}g_t^{\beta\lambda}q_t^{\alpha})\right] I\; G_5(\omega,\omega') \;, \tag{24}$$

$$\int d^4x\; e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha\beta}_{2,-,\frac{5}{2}}(x)J^{\dagger}_{0,+,\frac{1}{2}}(0)\right)|0\rangle = (q_t^{\alpha}q_t^{\beta} - \frac{1}{3}g_t^{\alpha\beta}q_t^2) I\; G_6(\omega,\omega') \;, \tag{25}$$

$$\int d^4x\, e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha\beta}_{2,-,\frac{5}{2}}(x)J^{\dagger\gamma}_{1,+,\frac{3}{2}}(0)\right)|0\rangle = \frac{1}{2}i(\epsilon^{\beta\gamma\mu\nu}q_t^{\alpha}+\epsilon^{\alpha\gamma\mu\nu}q_t^{\beta})q_{t\mu}v_\nu I\, G_7(\omega,\omega')\,, \tag{26}$$

$$\int d^4x\, e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha\beta}_{2,-,\frac{5}{2}}(x)J^{\dagger\lambda}_{1,-,\frac{3}{2}}(0)\right)|0\rangle = \Big[\frac{1}{2}(g_t^{\alpha\lambda}q_t^{\beta}+g_t^{\beta\lambda}q_t^{\alpha})-\frac{1}{3}g_t^{\alpha\beta}q_t^{\lambda}\Big]I\, G_8^p(\omega,\omega') + \Big[q_t^{\alpha}q_t^{\beta}q_t^{\lambda}-\frac{1}{6}q_t^2(g_t^{\alpha\lambda}q_t^{\beta}+g_t^{\beta\lambda}q_t^{\alpha}+\frac{4}{3}g_t^{\alpha\beta}q_t^{\lambda})\Big]I\, G_8^f(\omega,\omega')\,, \tag{27}$$

$$\int d^4x\, e^{ik\cdot x}\langle\pi(q)|T\left(J^{\alpha\beta}_{2,-,\frac{5}{2}}(x)J^{\dagger\mu\nu\sigma}_{3,-,\frac{5}{2}}(0)\right)|0\rangle = T^g I\, G_9^g(\omega,\omega')+T^f I\, G_9^f(\omega,\omega')+T^{p1}I\, G_9^{p1}(\omega,\omega') +T^{p2}I\, G_9^{p2}(\omega,\omega')\,, \tag{28}$$

for the $j_\ell^P=\frac{5}{2}^-$ doublet, where $k'=k+q$, $\omega=2v\cdot k$, $\omega'=2v\cdot k'$. Note that the two P wave couplings between $(2,-,\frac{5}{2})$ and $(3,-,\frac{5}{2})$ in eq. (28) are not independent and satisfy the relation $g_9^{p2}=-\frac{1}{3}g_9^{p1}$.

First let us consider the function $G_1(\omega,\omega')$ in eq. (20). As a function of two variables, it has the following pole terms from the double dispersion relation

$$\frac{f_{-,\frac{1}{2}}f_{-,\frac{3}{2}}g_1}{(2\bar\Lambda_{-,\frac{1}{2}}-\omega')(2\bar\Lambda_{-,\frac{3}{2}}-\omega)}+\frac{c}{2\bar\Lambda_{-,\frac{1}{2}}-\omega'}+\frac{c'}{2\bar\Lambda_{-,\frac{3}{2}}-\omega}\,,$$

where f_{P,j_ℓ} denotes the decay constant defined in eq. (1). $\bar\Lambda_{P,j_\ell}=m_{P,j_\ell}-m_Q$.

We calculate the correlator (20) on the light-cone to the leading order of $\mathcal{O}(1/m_Q)$. The expression for $G_1(\omega,\omega')$ reads

$$-\frac{\sqrt{6}}{8}i\int_0^\infty dt\int dx e^{ikx}\delta(x-vt)\mathrm{Tr}\Big[(\mathcal{D}^t_\alpha-\frac{1}{3}\gamma^t_\alpha\not{\mathcal{D}}^t)\times(1+\not v)\gamma_5\langle\pi(q)|u(0)\bar d(x)|0\rangle\Big]\,. \tag{29}$$

The pion (or K/η) distribution amplitudes are defined as the matrix elements of nonlocal operators between the vacuum and pion state. Up to twist four they are [20, 21]:

$$\langle\pi(q)|\bar d(x)\gamma_\mu\gamma_5 u(0)|0\rangle = -if_\pi q_\mu\int_0^1 du\, e^{iuqx}\Big[\varphi_\pi(u)+\frac{1}{16}m_\pi^2x^2A(u)\Big]-\frac{i}{2}f_\pi m_\pi^2\frac{x_\mu}{qx}\int_0^1 du\, e^{iuqx}B(u)\,,$$
$$\langle\pi(q)|\bar d(x)i\gamma_5 u(0)|0\rangle = f_\pi\mu_\pi\int_0^1 du\, e^{iuqx}\varphi_P(u)\,,$$
$$\langle\pi(q)|\bar d(x)\sigma_{\mu\nu}\gamma_5 u(0)|0\rangle = \frac{i}{6}(q_\mu x_\nu-q_\nu x_\mu)f_\pi\mu_\pi\times\int_0^1 du\, e^{iuqx}\varphi_\sigma(u)\,. \tag{30}$$

The expressions for the light cone wave functions $\varphi_\pi(u)$ etc are presented in Appendix B together with the relevant parameters for π, K and η.

Expressing eq. (29) with the light cone wave functions, we get the expression for the correlator function in the quark-gluon level

$$G_1(\omega,\omega') = -i\frac{\sqrt{6}}{12}f_\pi\int_0^\infty dt\int_0^1 du e^{i(1-u)\frac{\omega t}{2}}e^{iu\frac{\omega' t}{2}}u\times\Big\{\frac{i}{t}[u\varphi_\pi(u)]'+\frac{1}{16}m_\pi^2[uA(u)]'+\frac{1}{2}B(u)\Big[\frac{iu}{q\cdot v}-\frac{1}{(q\cdot v)^2t}\Big]+\mu_\pi u\varphi_p(u)+\frac{1}{6}\mu_\pi\varphi_\sigma(u)\Big\}+\cdots\,. \tag{31}$$

After performing wick rotation and double Borel transformation with the variables ω and ω' the single-pole terms in eq. (29) are eliminated and we arrive at the following result:

$$g_1 f_{-,\frac{1}{2}}f_{-,\frac{3}{2}} = \frac{\sqrt{6}}{6}f_\pi\exp\Big[\frac{\Lambda_{-,\frac{1}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\Big]\Big\{\frac{1}{2}[u\varphi_\pi(u)]'T^2f_1\Big(\frac{\omega_c}{T}\Big)-\frac{1}{8}m_\pi^2[uA(u)]'+m_\pi^2[G_1(u)+G_2(u)]-\mu_\pi[u\varphi_P(u)+\frac{1}{6}\varphi_\sigma(u)]Tf_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,, \tag{32}$$

where $u_0=\frac{T_1}{T_1+T_2}$, $T=\frac{T_1T_2}{T_1+T_2}$. T_1, T_2 are the Borel parameters, and $f_n(x)=1-e^{-x}\sum\limits_{k=0}^{n}\frac{x^k}{k!}$. The factor f_n is used to subtract the integral $\int_{\omega_c}^\infty s^ne^{-\frac{s}{T}}ds$ as a contribution of the continuum. The sum rules we have obtained from the correlators (20)-(28) are collected in Appendix C together with the definitions of G_1 etc.

IV. NUMERICAL RESULTS

For the ground states and P wave heavy mesons, we will use [17, 18]:

$$\bar\Lambda_{-,\frac{1}{2}}=0.5\ \mathrm{GeV}\,,\qquad f_{-,\frac{1}{2}}=0.25\ \mathrm{GeV}^{3/2}\,,$$
$$\bar\Lambda_{+,\frac{1}{2}}=0.85\ \mathrm{GeV}\,,\qquad f_{+,\frac{1}{2}}=0.36\pm0.10\ \mathrm{GeV}^{1/2}\,,$$
$$\bar\Lambda_{+,\frac{3}{2}}=0.95\ \mathrm{GeV}\,,\qquad f_{+,\frac{3}{2}}=0.26\pm0.06\ \mathrm{GeV}^{5/2}\,.$$

The mass parameters and decay constants for the D wave doublets have been obtained in section II from the two point sum rule:

$$\bar{\Lambda}_{-,\frac{3}{2}} = 1.42 \text{ GeV} , \quad f_{-,\frac{3}{2}} = 0.39 \pm 0.03 \text{ GeV}^{5/2} ,$$
$$\bar{\Lambda}_{-,\frac{5}{2}} = 1.38 \text{ GeV} , \quad f_{-,\frac{5}{2}} = 0.33 \pm 0.04 \text{ GeV}^{7/2} .$$

We choose to work at the symmetric point $T_1 = T_2 = 2T$, i.e., $u_0 = 1/2$ as traditionally done in literature [15]. The working region for T can be obtained by requiring that the higher twist contribution is less than 30% and the continuum contribution is less than 40% of the whole sum rule, we then get $\omega_c = 3.2-3.6$ GeV and the working region $2.0 < T < 2.5$ GeV for eqs. (C2), (C6) and (C12) in Appendix C and $1.2 < T < 2.0$ GeV for others. The working regions for the first three sum rules are higher than that for the others because there are zero points between 1 and 2 GeV for them and stability develops only for T above 2 GeV. From eq. (32) the coupling reads

$$g_{1\pi} f_{-,\frac{1}{2}} f_{-,\frac{3}{2}} = (0.17 \pm 0.04) \text{ GeV}^3 . \qquad (33)$$

We use the central values for the mass parameters and the error is due to the variation of T and the uncertainty of ω_c. The central value corresponds to $T = 1.6$ GeV and $\omega_c = 3.4$ GeV. There is cancelation between the twist 2 and twist 3 contributions in the sum rule.

For D wave heavy mesons with a strange quark, the couplings can be obtained by the same way. Notice that in the η case, f_π should be replaced by $-\frac{2}{\sqrt{6}} f_\eta$ due to the quark components of η meson, where $f_\eta = 0.16$ GeV is the decay constant of η meson. From eq. (32) we can get the couplings between the ground state doublet and D wave doublet with a strange quark,

$$g_{1K} f_{-,\frac{1}{2}} f_{-,\frac{3}{2}} = (0.19 \pm 0.06) \text{ GeV}^3 ,$$
$$g_{1\eta} f_{-,\frac{1}{2}} f_{-,\frac{3}{2}} = (0.28 \pm 0.06) \text{ GeV}^3 . \qquad (34)$$

The couplings between the $\frac{3}{2}^-$ and $\frac{5}{2}^-$ doublets and other doublets are collected in Table I. We can see that the $SU(3)_f$ breaking effect is not very big here.

$\frac{3}{2}^-$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+_d$	$\frac{3}{2}^+_s$	$\frac{3}{2}^-_f$	$\frac{3}{2}^-_p$
π	0.17	0.086	0.16	0.10	0.056	0.071
K	0.19	0.09	0.24	0.18	0.057	0.10
η	0.28	0.046	0.22	0.11	0.030	0.078

$\frac{5}{2}^-$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-_f$	$\frac{3}{2}^-_p$	$\frac{5}{2}^-_g$	$\frac{5}{2}^-_f$	$\frac{5}{2}^-_p$
π	0.11	0.36	0.072	0.13	0.12	0.015	0.05	0.01
K	0.14	0.48	0.083	0.11	0.16	0.015	0.09	0.02
η	0.12	0.42	0.074	0.10	0.14	0.008	0.08	0.01

TABLE I: The pionic couplings between $\frac{3}{2}^-$ and $\frac{5}{2}^-$ doublets and other doublets. The values are the product of coupling constants and the decay constants of initial and final heavy mesons.

With the central values of f's, we get the absolute values of the coupling constants:

$$g_{1\pi} = (1.74 \pm 0.43) \text{ GeV}^{-1} ,$$
$$g_{1K} = (1.95 \pm 0.63) \text{ GeV}^{-1} ,$$
$$g_{1\eta} = (2.87 \pm 0.65) \text{ GeV}^{-1} . \qquad (35)$$

For the $\frac{5}{2}^-$ doublet we have

$$g_{5\pi} = (1.33 \pm 0.29) \text{ GeV}^{-3} ,$$
$$g_{5K} = (1.70 \pm 0.42) \text{ GeV}^{-3} ,$$
$$g_{5\eta} = (1.45 \pm 0.30) \text{ GeV}^{-3} . \qquad (36)$$

We do not include the uncertainties due to f's here.

We can also extract the mass parameter from the strong coupling formulas obtained in the last section. By putting the exponential factor on the left side of eq. (C6) and differentiating it, one obtains

$$\bar{\Lambda}_{-,\frac{3}{2}} = \frac{T^2}{2} \frac{\mathrm{d}[\varphi_\pi(u_0) T f_0(\frac{\omega_c}{T}) - \frac{1}{4} m_\pi^2 A(u_0) \frac{1}{T}]/\mathrm{dT}}{[\varphi_\pi(u_0) T f_0(\frac{\omega_c}{T}) - \frac{1}{4} m_\pi^2 A(u_0) \frac{1}{T} - \frac{1}{3} \mu_\pi \varphi_\sigma(u_0)]} . \qquad (37)$$

With $\omega_c = 3.2-3.6$ and the working region $2.0 < T < 2.5$ GeV, we get

$$\bar{\Lambda}_{1,-,\frac{3}{2}} = 1.36 - 1.56 \text{ GeV} , \qquad (38)$$

which is consistent with the value obtained by two point sum rule. We present the variation of mass with T and ω_c in Fig. 3.

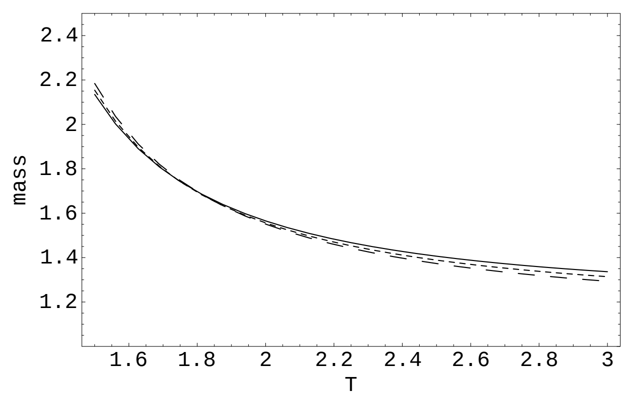

FIG. 3: The variation of the mass parameter $\bar{\Lambda}_{-,\frac{3}{2}}$ with the threshold ω_c (in unit of GeV) for the $\frac{3}{2}^-$ doublet from LCQSR. The long-dashed, short dashed and solid curves correspond to ω_c=3.2, 3.4, 3.6 GeV respectively.

V. STRONG DECAY WIDTHS FOR D WAVE HEAVY MESONS

Having calculated the coupling constants, one can obtain the pionic decay widths of D wave heavy mesons.

The widths for D wave states decaying to 0^-, 1^-, 1^+ states are

$$\Gamma(B_1^{*'} \to B^0\pi^-) = \frac{1}{24\pi}\frac{M_B}{M_{B_1^{*'}}} g_1^2 |\boldsymbol{p}_1|^3 ,$$

$$\Gamma(B_1^{*'} \to B^{*0}\pi^-) = \frac{1}{12\pi}\frac{M_{B^*}}{M_{B_1^{*'}}} g_1^2 |\boldsymbol{p}_1|^3 ,$$

$$\Gamma(B_1^{*'} \to B_1^0\pi^-) = \frac{1}{36\pi}\frac{M_{B_1}}{M_{B_1^{*'}}} g_2^2 |\boldsymbol{p}_1|^5 ,$$

$$\Gamma(B_2^{*} \to B^{*0}\pi^-) = \frac{1}{36\pi}\frac{M_{B^*}}{M_{B_2}} g_1^2 |\boldsymbol{p}_1|^3 ,$$

$$\Gamma(B_3 \to B^0\pi^-) = \frac{1}{140\pi}\frac{M_B}{M_{B_3}} g_5^2 |\boldsymbol{p}_1|^7 ,$$

$$\Gamma(B_3 \to B^{*0}\pi^-) = \frac{1}{105\pi}\frac{M_{B^*}}{M_{B_3}} g_5^2 |\boldsymbol{p}_1|^7 , \qquad (39)$$

where $|\boldsymbol{p}_1|$ is the moment of final state π. Note that $g(B_2^* B^*) = \sqrt{\frac{2}{3}}\, g(B_1^{*'} B^*) = \sqrt{\frac{2}{3}}\, g_1$.

A. Nonstrange case

We take 2.8 GeV and 6.2 GeV for the masses of D wave charmed and bottomed mesons respectively. $M_D = 1.87$ GeV, $M_{D^*} = 2.01$ GeV, $M_{D_1} = 2.42$ GeV, $M_B = 5.28$ GeV, $M_{B^*} = 5.33$ GeV [19] and $M_{B_1} = 5.75$ GeV from quark model prediction [10]. After summing over the charged and neutral modes, we get the results listed in Table II.

	$D\pi$	$D^*\pi$	$D_1\pi$		$D^*\pi$
$D_1^{*'} \to$	9-27	13-39	0.2	$D_2^* \to$	5-13
	$B\pi$	$B^*\pi$	$B_1\pi$		$B^*\pi$
$B_1^{*'} \to$	16-46	27-79	0.3	$B_2^* \to$	9-27

TABLE II: The decay widths (in unit MeV) of the charmed and bottomed D wave $(1^-, 2^-)$ to ground doublets and π.

B. Strange case

We use $M_{D_s} = 1.97$ GeV, $M_{D_s^*} = 2.11$ GeV, $M_{B_s} = 5.37$ GeV, $M_{B_s^*} = 5.41$ GeV [19]. Then for the charm-strange sector we have Table III. We do not consider the DK^* mode and three-body modes in the present work.

For the 2^-, 3^- states with $j_\ell = \frac{5}{2}$, we find that the widths are quite small and the branching fraction is perhaps more useful. In the charm-strange sector the ratio of widths (central values) for DK, D^*K, $D_s\eta$ and $D^*\eta$ modes is 1 : 0.4 : 0.1 : 0.02.

	DK	D^*K	$D_s\eta$	$D_s^*\eta$
$D_{s1}^{*'} \to$	8-28	10-48	8-22	8-20
$D_{s2}^{*} \to$		4-16		3-7
	BK	B^*K	$B_s\eta$	$B_s^*\eta$
$B_{s1}^{*'} \to$	12-52	18-84	11-27	16-42
$B_{s2}^{*} \to$		6-28		6-14

TABLE III: The decay widths (in unit MeV) of the charm-strange and bottom-strange D wave $(1^-, 2^-)$ to ground doublets and K/η.

VI. CONCLUSION

In this work we extract the mass and decay constants using the traditional two point sum rule and calculate the strong couplings of D wave heavy meson doublets with light hadrons π, K and η using LCQSR in the leading order of HQET. We also extract the mass parameter from LCQSR for the coupling within the same D wave doublet. The extracted mass parameters from two approaches are consistent with each other. We then calculate the widths of D wave heavy mesons decaying to light hadrons.

We have not considered the $1/m_Q$ correction and radiative corrections. Heavy quark expansion works well for B mesons where the $1/m_b$ correction is under control and not so large. However, the $1/m_c$ correction is not so small for the charmed mesons. It will be desirable to consider both the $1/m_Q$ and radiative corrections in the future investigation.

According to our present calculation, the ratios such as $\frac{\Gamma(D_{sJ}(2860)\to DK)}{\Gamma(D_{sJ}(2860)\to D_s\eta)}$ are useful in distinguishing various interpretations of $D_{sJ}(2860)$ and $D_{sJ}(2715)$. Treating $D_{sJ}(2860)$ as a D wave 1^- state, we find the above ratio is $0.4 - 2.2$. If it is the radial excitation of D_s^*, this ratio is 0.09 [8].

The pionic widths of D wave states are not very large. With a mass of 2.86 GeV, the partial decay width of the 1^- D wave D_s state into DK and $D\eta$ modes is $34 - 118$ MeV. With a mass of 2.715 GeV its pionic width is $15 - 57$ MeV. Note that DK^* modes may be equally important. So detection of other decay channels, such as $D_s\eta$ and D^*K modes, will be very helpful in the classification of these new states.

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APPENDIX A

The $T^{\alpha,\beta;\,\mu,\nu}$ and $T^{\alpha,\beta,\lambda;\,\mu,\nu,\sigma}$ are defined as

$$T^{\alpha,\beta;\,\mu,\nu} = \frac{1}{2}(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha,\nu}g^{\beta,\mu})-\frac{1}{3}g_t^{\alpha\beta}g_t^{\mu\nu}\,,$$

$$\begin{aligned}T^{\alpha,\beta,\lambda;\,\mu,\nu,\sigma} &= \frac{1}{6}(g^{\alpha\mu}g^{\beta\nu}g^{\lambda\sigma}+g^{\alpha\mu}g^{\beta\sigma}g^{\lambda\beta}+g^{\alpha\nu}g^{\beta\mu}g^{\lambda\sigma}\\ &+ g^{\alpha\nu}g^{\beta\sigma}g^{\lambda\mu}+g^{\alpha\sigma}g^{\beta\mu}g^{\lambda\nu}+g^{\alpha\sigma}g^{\beta\nu}g^{\lambda\mu})\\ &- \frac{1}{9}(g_t^{\alpha\beta}g_t^{\mu\nu}g_t^{\lambda\sigma}+g_t^{\alpha\lambda}g_t^{\mu\nu}g_t^{\beta\sigma}+g_t^{\beta\lambda}g_t^{\mu\nu}g_t^{\alpha\sigma})\,.\end{aligned}$$

The tensor structures for G wave, F wave and two P wave decays in eq. (28) for the coupling between the two $\frac{5}{2}^-$ state are

$$\begin{aligned}T^g &= T^{\mu,\nu,\sigma;\mu_1,\nu_1,\sigma_1}T^{\alpha,\beta;\alpha_1,\beta_1}q_{\mu_1}q_{\nu_1}q_{\sigma_1}q_{\alpha_1}q_{\beta_1}\,,\\ T^f &= \frac{1}{3}\Big\{\Big[q_t^\mu q_t^\alpha q_t^\beta-\frac{1}{6}q_t^2(g_t^{\mu\alpha}q_t^\beta+g_t^{\mu\beta}q_t^\alpha\\ &+\frac{4}{3}g_t^{\alpha\beta}q_t^\mu)\Big]g_t^{\nu\sigma}+(\mu,\nu,\sigma)\Big\}\,,\\ T^{p1} &= \frac{1}{6}\Big[g_t^{\mu\alpha}g_t^{\nu\sigma}q_t^\beta+g_t^{\mu\beta}g_t^{\nu\sigma}q_t^\alpha+(\mu,\nu,\sigma)\Big]\,,\\ T^{p2} &= \frac{1}{3}(q_t^\mu g_t^{\nu\sigma}+q_t^\nu g_t^{\mu\sigma}+q_t^\sigma g_t^{\mu\nu})g_t^{\alpha\beta}\,.\end{aligned}$$

APPENDIX B

The distribution amplitudes φ_π etc can be parameterized as [20, 21]

$$\begin{aligned}\varphi_\pi(u) &= 6u\bar{u}\Big[1+a_1C_1^{3/2}(\zeta)+a_2C_2^{3/2}(\zeta)\Big]\,,\\ \phi_p(u) &= 1+\Big[30\eta_3-\frac{5}{2}\rho_\eta^2\Big]C_2^{1/2}(\zeta)+\Big[-3\eta_3\omega_3\\ &-\frac{27}{20}\rho_\eta^2-\frac{81}{10}\rho_\eta^2a_2\Big]C_4^{1/2}(\zeta)\,,\\ \phi_\sigma(u) &= 6u(1-u)\Big\{1+\Big(5\eta_3-\frac{1}{2}\eta_3\omega_3-\frac{27}{20}\rho_\eta^2\\ &-\frac{3}{5}\rho_\eta^2a_2\Big)\Big\}C_2^{3/2}(\zeta)\,,\\ g_\pi(u) &= 1+\Big[1+\frac{18}{7}a_2+60\eta_3+\frac{20}{3}\eta_4\Big]C_2^{1/2}(\zeta)\\ &+\Big[-\frac{9}{28}a_2-6\eta_3\omega_3\Big]C_4^{1/2}(\zeta)\,,\\ A(u) &= 6u\bar{u}\Big\{\frac{16}{15}+\frac{24}{35}a_2+20\eta_3+\frac{20}{9}\eta_4+\Big[-\frac{1}{15}+\frac{1}{16}\\ &-\frac{7}{27}\eta_3\omega_3-\frac{10}{27}\eta_4\Big]C_2^{3/2}(\zeta)+\Big[-\frac{11}{210}a_2\\ &-\frac{4}{135}\eta_3\omega_3\Big]C_4^{3/2}(\zeta)\Big\}+\Big(-\frac{18}{5}a_2+21\eta_4\omega_4\Big)\\ &\Big\{2u^3(10-15u+6u^2)\ln u+2\bar{u}^3(10-15\bar{u}\\ &+6\bar{u}^2)\ln\bar{u}+u\bar{u}(2+13u\bar{u})\Big\}\,,\end{aligned} \tag{B1}$$

where $\bar{u}\equiv 1-u$, $\zeta\equiv 2u-1$. $C_{1,2}^{3/2,1/2}(\zeta)$ are Gegenbauer polynomials. Here $g_\pi(u)=B(u)+\varphi_\pi(u)$. $a_1^{\pi,\eta}=0, a_1^K=0.06$, $a_2^{\pi,K,\eta}=0.25$, $\eta_3^{\pi,K}=0.015$, $\eta_3^\eta=0.013$, $\omega_3^{\pi,K,\eta}=-3$, $\eta_4^\pi=10$, $\eta_4^K=0.6$, $\eta_4^\eta=0.5$, $\omega_4^{\pi,K,\eta}=0.2$. ρ_π^2 etc give the mass corrections and are defined as $\rho_\pi^2=\frac{(m_u+m_d)^2}{m_\pi^2}$, $\rho_K^2=\frac{m_s^2}{m_K^2}$, $\rho_\eta^2=\frac{m_s^2}{m_\eta^2}$. $m_s=0.125$ GeV. $\mu_\pi=\frac{m_\pi^2}{m_u+m_d}(1-\rho_\pi^2)$, $\mu_{K,\eta}=\frac{m_{K,\eta}^2}{m_s}(1-\rho_{K,\eta}^2)$. $f_\pi=0.13$ GeV, $f_K=0.16$ GeV, $f_\eta=0.156$ GeV. All of them are scaled at $\mu=1$ GeV.

APPENDIX C

In this appendix we collect the sum rules we have obtained for the strong couplings of D wave heavy doublets with light hadrons.

$$\begin{aligned}g_1f_{-,\frac{1}{2}}f_{-,\frac{3}{2}} &= \frac{\sqrt{6}}{6}f_\pi\exp\Big[\frac{\Lambda_{-,\frac{1}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\Big]\Big\{\frac{1}{2}[u\varphi_\pi(u)]'T^2f_1\Big(\frac{\omega_c}{T}\Big)-\frac{1}{8}m_\pi^2[uA(u)]'+m_\pi^2[G_1(u)+G_2(u)]\\ &-\mu_\pi\Big[u\varphi_P(u)+\frac{1}{6}\varphi_\sigma(u)\Big]Tf_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,,\end{aligned} \tag{C1}$$

$$g_2f_{+,\frac{1}{2}}f_{-,\frac{3}{2}} = \frac{\sqrt{6}}{4}f_\pi\exp\Big[\frac{\Lambda_{+,\frac{1}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\Big]u\Big\{\varphi_\pi(u)Tf_0\Big(\frac{\omega_c}{T}\Big)-\frac{1}{4}m_\pi^2A(u)\frac{1}{T}-\frac{1}{3}\mu_\pi\varphi_\sigma(u)\Big\}\Big|_{u=u_0}\,, \tag{C2}$$

$$\begin{aligned}g_3^df_{+,\frac{3}{2}}f_{-,\frac{3}{2}} &= \frac{1}{8}f_\pi\exp\Big[\frac{\Lambda_{+,\frac{3}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\Big]\Big\{[(u(1-u)\varphi_\pi(u)]'T^2f_1\Big(\frac{\omega_c}{T}\Big)-\frac{1}{4}m_\pi^2[(u(1-u)A(u)]'\\ &+2m_\pi^2\big[G_3(u)+G_4(u)+2G_5(u)\big]+2\mu_\pi\big[u(1-u)\varphi_p(u)+\frac{1}{6}\varphi_\sigma(u)\big]Tf_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,,\end{aligned} \tag{C3}$$

$$
\begin{aligned}
g_3^s f_{+,\frac{3}{2}} f_{-,\frac{3}{2}} \;=\;& -\frac{1}{48} f_\pi \exp\left[\frac{\Lambda_{+,\frac{3}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\right]\Big\{\big[(u(1-u)\varphi_\pi(u)\big]^{'''} T^4 f_3\Big(\frac{\omega_c}{T}\Big) + 2\mu_\pi \left[\Big(u(1-u)\varphi_p(u)\Big)^{''}\right. \\
& \left. +\frac{1}{6}\varphi_\sigma(u)^{''}\right] T^3 f_2\Big(\frac{\omega_c}{T}\Big) - m_\pi^2 \Big[4\Big(u(1-u)\varphi_\pi(u)\Big)^{'} + \frac{1}{4}\Big(u(1-u)A(u)\Big)^{'''} \\
& +\frac{3}{2}A(u)^{'} - 2(1-2u)B(u) - 2\Big(u(1-u)B(u)\Big)^{'} - 4G_6(u)\Big] T^2 f_1\Big(\frac{\omega_c}{T}\Big) \\
& -8m_\pi^2\mu_\pi\,\Big[u(1-u)\varphi_p(u)+\frac{1}{6}\varphi_\sigma(u)\Big]\,T f_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,,
\end{aligned}
\tag{C4}
$$

$$
g_4^p f_{-,\frac{3}{2}} f_{-,\frac{3}{2}} \;=\; \frac{\sqrt{6}}{96} f_\pi \exp\left[\frac{\Lambda_{-,\frac{3}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\right]\Big\{m_\pi^2 A(u) T f_0\Big(\frac{\omega_c}{T}\Big) - \frac{2}{3}\mu_\pi\big[(1-u)\varphi_\sigma(u)\big]^{'} T^2 f_1\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,, \tag{C5}
$$

$$
g_4^f f_{-,\frac{3}{2}} f_{-,\frac{3}{2}} \;=\; \frac{\sqrt{6}}{4} f_\pi \exp\left[\frac{\Lambda_{-,\frac{3}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\right] u(1-u)\Big\{\varphi_\pi(u) T f_0\Big(\frac{\omega_c}{T}\Big) - \frac{1}{4T} m_\pi^2 A(u) - \frac{1}{3}\mu_\pi\varphi_\sigma(u)\Big\}\Big|_{u=u_0}\,, \tag{C6}
$$

$$
g_5 f_{-,\frac{1}{2}} f_{-,\frac{5}{2}} \;=\; \frac{1}{2} f_\pi \exp\left[\frac{\Lambda_{-,\frac{3}{2}}+\Lambda_{-,\frac{3}{2}}}{T}\right] u^2\Big\{\varphi_\pi(u) T f_0\Big(\frac{\omega_c}{T}\Big) - \frac{1}{4T} m_\pi^2 A(u) + \frac{1}{3}\mu_\pi\varphi_\sigma(u)\Big\}\Big|_{u=u_0}\,, \tag{C7}
$$

$$
\begin{aligned}
g_6 f_{+,\frac{1}{2}} f_{-,\frac{5}{2}} \;=\;& -\frac{\sqrt{15}}{20} f_\pi \exp\left[\frac{\Lambda_{+,\frac{1}{2}}+\Lambda_{-,\frac{5}{2}}}{T}\right]\Big\{\big[u^2\varphi_\pi(u)\big]^{'} T^2 f_1\Big(\frac{\omega_c}{T}\Big) - \frac{1}{4} m_\pi^2\big[u^2 A(u)\big]^{'} \\
& -\frac{1}{2} m_\pi^2\big[G_7(u)+2G_8(u)+2G_5(u)\big] + 2\mu_\pi u\Big[u\varphi_p(u)+\frac{1}{3}\varphi_\sigma(u)\Big] T f_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,,
\end{aligned}
\tag{C8}
$$

$$
\begin{aligned}
g_7 f_{+,\frac{3}{2}} f_{-,\frac{5}{2}} \;=\;& -\frac{\sqrt{10}}{30} f_\pi \exp\left[\frac{\Lambda_{+,\frac{3}{2}}+\Lambda_{-,\frac{5}{2}}}{T}\right]\Big\{-\frac{1}{4}\big[u^2(1-u)\varphi_\pi(u)\big]^{''} T^3 f_2\Big(\frac{\omega_c}{T}\Big) - \frac{1}{12}\mu_\pi\Big[7\Big(u(1-\frac{2}{7}u)\varphi_\sigma(u)\Big)^{'} \\
& +\Big(u^2(1-u)\varphi_\sigma(u)\Big)^{''}\Big] T^2 f_1\Big(\frac{\omega_c}{T}\Big) + m_\pi^2\Big[u^2(1-u)\varphi_\pi(u) + \frac{7}{8}uA(u) \\
& +\frac{1}{16}\Big(u^2(1-u)A(u)\Big)^{''}\Big] T f_0\Big(\frac{\omega_c}{T}\Big) + \frac{1}{3} m_\pi^2\mu_\pi u^2(1-u)\varphi_\sigma(u)\ \Big\}\Big|_{u=u_0}\,,
\end{aligned}
\tag{C9}
$$

$$
\begin{aligned}
g_8^p f_{-,\frac{3}{2}} f_{-,\frac{5}{2}} \;=\;& \frac{\sqrt{10}}{18} f_\pi \exp\left[\frac{\Lambda_{-,\frac{3}{2}}+\Lambda_{-,\frac{5}{2}}}{T}\right]\Big\{\frac{1}{8}\big[u^2(1-u)\varphi_\pi(u)\big]^{'''} T^4 f_3\Big(\frac{\omega_c}{T}\Big) + \frac{1}{4}\mu_\pi\Big[\Big(u^2(1-u)\varphi_p(u)\Big)^{''} \\
& +\frac{1}{3}\Big(u(1-\frac{5}{2}u)\varphi_\sigma(u)\Big)^{''}\Big] T^3 f_2\Big(\frac{\omega_c}{T}\Big) - \frac{1}{2} m_\pi^2\Big[\Big(u^2(1-u)\varphi_\pi(u)\Big)^{'} + \frac{27}{40}\Big(uA(u)\Big)^{'} \\
& +\frac{1}{16}\Big(u^2(1-u)A(u)\Big)^{'''} - \frac{1}{5}u(1-\frac{3}{2}u)B(u) - \frac{1}{2}\Big(u^2(1-u)B(u)\Big)^{'} - G_9(u) \\
& +\frac{9}{5}G_2(u)\Big] T^2 f_1\Big(\frac{\omega_c}{T}\Big) - m_\pi^2\mu_\pi\Big[u^2(1-u)\varphi_p(u) + \frac{1}{3}(1-\frac{5}{2}u)\varphi_\sigma(u)\Big] T f_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,,
\end{aligned}
\tag{C10}
$$

$$
\begin{aligned}
g_8^f f_{-,\frac{3}{2}} f_{-,\frac{5}{2}} \;=\;& \frac{\sqrt{10}}{15} f_\pi \exp\left[\frac{\Lambda_{1,-,\frac{3}{2}}+\Lambda_{2,-,\frac{5}{2}}}{T}\right]\Big\{-\frac{1}{2}\big[u^2(1-u)\varphi_\pi(u)\big]^{'} T^2 f_1\Big(\frac{\omega_c}{T}\Big) + \frac{1}{8} m_\pi^2\big[u^2(1-u)A(u)\big]^{'} \\
& +m_\pi^2\big[G_{10}(u) - 2G_{11}(u) + 2G_{12}(u) - 6G_{13}(u)\big] \\
& -\mu_\pi u(1-u)\Big[u\varphi_p(u) + \frac{1}{12}\varphi_\sigma(u)\Big] T f_0\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}\,,
\end{aligned}
\tag{C11}
$$

$$g_9^g f^2_{-,\frac{5}{2}} = \frac{\sqrt{15}}{6} f_\pi \exp\Big[\frac{\Lambda_{-,\frac{5}{2}}+\Lambda_{-,\frac{5}{2}}}{T}\Big] u^2(1-u)^2 \Big\{\varphi_\pi(u) T f_0\Big(\frac{\omega_c}{T}\Big) - \frac{1}{4} m_\pi^2 A(u) \frac{1}{T} - \frac{1}{3}\mu_\pi \varphi_\sigma(u)\Big\}\Big|_{u=u_0}, \tag{C12}$$

$$\begin{aligned} g_9^f f^2_{-,\frac{5}{2}} = {} & \frac{\sqrt{15}}{45} f_\pi \exp\Big[\frac{\Lambda_{-,\frac{5}{2}}+\Lambda_{-,\frac{5}{2}}}{T}\Big]\Big\{ -\frac{1}{4}\big[u^2(1-u)^2\varphi_\pi(u)\big]'' T^3 f_2\Big(\frac{\omega_c}{T}\Big) + \frac{1}{12}\mu_\pi \Big[\big(u^2(1-u)^2\varphi_\sigma(u)\big)'' , \\ & -\frac{3}{8}\big(u(1-u)^2\varphi_\sigma(u)\big)'\Big] T^2 f_1\Big(\frac{\omega_c}{T}\Big) + m_\pi^2\Big[u^2(1-u)^2\varphi_\pi(u) - \frac{1}{8}u(3+7u)A(u) \\ & -3\big(G_{10}(u)+2G_{11}(u)+2G_{12}(u)-6G_{13}(u)\big)\Big] T f_0\Big(\frac{\omega_c}{T}\Big) - \frac{1}{3}m_\pi^2\mu_\pi u^2(1-u)^2\varphi_\sigma(u)\Big\}\Big|_{u=u_0}, \end{aligned} \tag{C13}$$

$$\begin{aligned} g_9^{p1} f^2_{-,\frac{5}{2}} = {} & -\frac{\sqrt{15}}{45} f_\pi \exp\Big[\frac{\Lambda_{2,-,\frac{5}{2}}+\Lambda_{3,-,\frac{5}{2}}}{T}\Big]\Big\{\frac{1}{24}\mu_\pi\big[u^2(1-u)\varphi_\sigma(u)\big]''' T^4 f_3\Big(\frac{\omega_c}{T}\Big) \\ & -\frac{1}{8}m_\pi^2\Big[\frac{3}{2}\big(u(1-2u)A(u)\big)'' + u(2-3u)B(u) + \big(u^2(1-u)B(u)\big)' \\ & +2G_9(u) - 6G_2(u)\Big] T^3 f_2\Big(\frac{\omega_c}{T}\Big) - \frac{1}{6}m_\pi^2\mu_\pi\big[u^2(1-u)^2\varphi_\sigma(u)\big]' T^2 f_1\Big(\frac{\omega_c}{T}\Big)\Big\}\Big|_{u=u_0}, \end{aligned} \tag{C14}$$

The G's are defined as integrals of light cone wave function $B(u)$

$$\begin{aligned} & G_1(u) \equiv \int_0^u tB(t)dt, \quad G_2(u) \equiv \int_0^u dx \int_0^x B(t)dt \,, \\ & G_3(u) \equiv \int_0^u t(1-t)B(t)dt, \quad G_4(u) \equiv \int_0^u dx \int_0^x (1-2t)B(t)dt \,, \\ & G_5(u) \equiv \int_0^u dx \int_0^x dy \int_0^y B(t)dt \,, \quad G_6(u) \equiv \int_0^u B(t)dt \,, \\ & G_7(u) \equiv \int_0^u t^2 B(t)dt \,, \quad G_8(u) \equiv \int_0^u dx \int_0^x tB(t)dt \,, \\ & G_9(u) \equiv \int_0^u (1-3t)B(t)dt \,, \quad G_{10}(u) \equiv \int_0^u t^2(1-t)B(t)dt \,, \\ & G_{11}(u) \equiv \int_0^u dx \int_0^x t(1-\frac{3}{2}t)B(t)dt \,, \quad G_{12}(u) \equiv \int_0^u dx \int_0^x dy \int_0^y (1-3t)B(t)dt \,, \\ & G_{13}(u) \equiv \int_0^u dx \int_0^x dy \int_0^y dz \int_0^z B(t)dt \,. \end{aligned}$$

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